Lagrangian mechanics on manifolds

- Differentiable manifolds (manifolds)

  - A manifold is the natural notion of a smooth object.
    Example: $\mathbb{R}^2, \mathbb{R}^3$. n-dimensional sphere
    $S^n = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} ; |x|^2 = 1\}$

  - Locally "looks like" $\mathbb{R}^n$
    Example: A cube is not a manifold "corners"
    $SO(n)$ is a manifold.

- $F : \mathbb{R}^n \to \mathbb{R}^m$ $F(x) = (F_1(x) \dots F_m(x))$, $x \in \mathbb{R}^n$
  $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ $\frac{\partial F_i}{\partial x_j}(p)$, $\frac{|F(p+h) - F(p) - dF_p(h)|}{|h|} \to 0$ as $|h| \to 0$

  Regular value theorem: Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map and for all $p \in F^{-1}(c)$, where
  $$F^{-1}(c) = \{p \in \mathbb{R}^{n+m} : F(p) = c\} \neq \phi$$
  the derivative $dF_p : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective
  Then $F^{-1}(c)$ is an n-dimensional manifold.

- Formal definition.

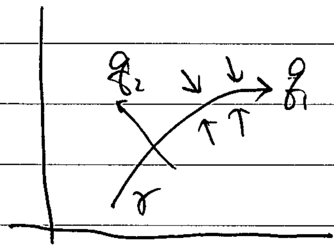
Def: An n-dimensional manifold is a topological space M s.t. $\exists$ a family $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ where
① U_i is an open set in M and $\bigcup_{i \in I} U_i = M$
② $\varphi_i : U_i \to \mathbb{R}^n$ is a continuous bijection
homeomorphism
③ If $U_i \cap U_j \neq \phi$, $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth bijection with smooth inverse.

$\mathcal{A}$: atlas $\qquad (U_i, \varphi_i)$ charts

Example: $\mathbb{R}^n$: $\quad U = \mathbb{R}^n.\ \varphi = id.\ \ \mathcal{A} = \{(U, \varphi)\}$

Lagrangian mechanics on manifolds

- Holonomic constraints 完整約束



γ: smooth curve in the plane

$$\begin{cases} U_N = N q_2^2 + U_0(q_1, q_2) \\ q_1(0) = q_1^0, \ \dot{q}_1(0) = \dot{q}_1^0, \ q_2(0) = 0, \ \dot{q}_2(0) = 0 \end{cases}$$

Let $q_1 = \varphi(t, N)$

Thm: $N \to \infty$, $\lim\limits_{N\to\infty} \varphi(t, N) = \psi(t)$, $q_1 = \psi(t)$ satisfies

$$\frac{d}{dt}\left(\frac{\partial L_*}{\partial \dot{q}_1}\right) = \frac{\partial L_*}{\partial q_1}, \quad \text{where } L_*(q_1, \dot{q}_1) = T\big|_{q_2 = \dot{q}_2 = 0} - U_0\big|_{q_2 = 0}$$

- Definition of a system with constraints

γ: m-dimensional surface in 3n dimensional configuration space. points $\vec{r}_1, \cdots \vec{r}_n$ with masses $m_1 \cdots m_n$

Let $\vec{q} = (q_1, \cdots q_m)$ be some coordinates on γ : $\vec{r}_i = \vec{r}_i(\vec{q})$

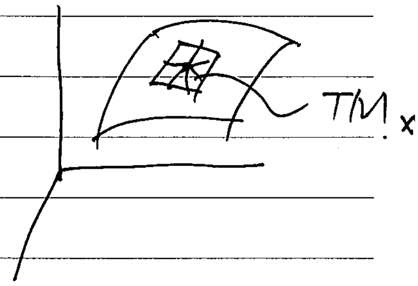
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial L}{\partial \vec{q}}, \quad L = \frac{1}{2}\sum m_i \dot{\vec{r}}_i^{\,2} + U(\vec{q})$$

is called a system of n points with 3n-m ideal holonomic constraints. $\gamma: f_1(\vec{r}) = 0 \cdots f_k(\vec{r}) = 0$, $k = 3n - m$

- Tangent space

M: k- dimensional manifold embedded in $\mathbb{E}^n$
tangent space TM_x is orthogonal to $\{grad f_1 \cdots grad f_{n-k}\}$.

$\dot{x} = \lim_{t \to 0} \frac{\vec{\varphi}(t) - \vec{\varphi}(0)}{t}$, where
$\vec{\varphi}(0) = \vec{x}$, $\vec{\varphi}(t) \in M$.



- Tangent bundle 切丛

The union of the tangent spaces to M: $\cup_{\vec{x} \in M} TM_{\vec{x}}$ has a natural differentiable manifold structure
The dimension is twice the dimension of M.

TM: tangent bundle $(q_1, \cdots q_n, \underbrace{\xi_1 \cdots \xi_n}_{dq_i})$

$p: TM \to M$ takes a tangent vector ξ to point $\vec{x} \in M$.
$(\xi \in TM_x)$ is the natural projection.

The inverse image $p^{-1}(\vec{x})$ is the tangent space $TM_{\vec{x}}$.

- Riemannian manifold : a manifold possessing a metric $\langle \xi, \xi \rangle$ on every tangent space TM_x.
$\uparrow$
positive definite.

Theorem: The evolution of local coordinates $\vec{q} = (q_1 \cdots q_n)$ of a point $\gamma(t)$ satisfies the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial L}{\partial \vec{q}},$$

where $L(\vec{q}, \dot{\vec{q}})$ is $L: TM \to \mathbb{R}$ in the coordinates $\vec{q}$ and $\dot{\vec{q}}$ on TM.

- For Riemannian manifold M, $T = \frac{1}{2}\langle \vec{v}, \vec{v} \rangle$, $\vec{v} \in TM_{\vec{x}}$

kinetic energy

potential energy : $U: M \to \mathbb{R}$

natural system: $L = T - U$

Noether's theorem.

Let M be a smooth manifold, $L: TM \to \mathbb{R}$ smooth function on its tangent bundle TM. $h: M \to M$ smooth map

Def: (M, L) admits the mapping h if for $\forall\ \vec{v} \in TM$,
$L(h_* \vec{v}) = L(\vec{v})$

Example: $M = \{(x_1, x_2, x_3)\}$, $L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U(x_2, x_3)$
$h: (x_1, x_2, x_3) \to (x_1 + s, x_2, x_3)$

Noether's theorem: If (M, L) admits the one-parameter group of diffeomorphism $h^s: M \to M$, $s \in \mathbb{R}$, then the Lagrangian system of equations corresponding to L has a first integral $I: TM \to \mathbb{R}$

$$I(\dot{\vec{q}}, \vec{q}) = \frac{\partial L}{\partial \dot{q}} \frac{d\, h^s(\vec{q})}{ds}\Big|_{s=0}$$

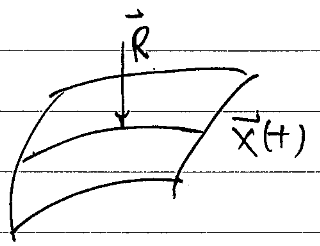
· D'Alembert's principle

完整约束 holonomic system 的几何描述

holonomic system (M, L), $M \sim$ surface in $\mathbb{R}^3$

$L = \frac{1}{2} m \dot{\vec{x}}^2 - U(\vec{x})$ $\vec{x}$ on the smooth surface M

Def: $\vec{R} = m\ddot{\vec{x}} + \frac{\partial U}{\partial \vec{x}}$ constraint force



$\Rightarrow$ Newton's eq.

$$m\ddot{\vec{x}} = -\frac{\partial U}{\partial \vec{x}} + \vec{R}$$

physical meaning of $\vec{R}$: $U + NU_1$, $U_1(\vec{x}) = \rho^2(\vec{x}, \vec{M})$

$N \to \infty$ $\vec{F} = -N\frac{\partial U_1}{\partial \vec{x}} \longrightarrow \vec{R}$

$(\vec{R}, \vec{\xi}) = 0$. for $\vec{\xi}$ tangent vector.

D'Alembert's principle:

$$\left(m\ddot{\vec{x}} + \frac{\partial U}{\partial \vec{x}}, \vec{\xi}\right) = 0$$ for any virtual variation $\vec{\xi}$
tangent vectors to the manifold

For a system of points $\vec{x}_i$ with m_i, $\vec{R}_i$ are defined by $\vec{R}_i = m_i\ddot{\vec{x}}_i + (\partial U / \partial \vec{x}_i)$

D'Alembert's principle $\Rightarrow$ $\sum(\vec{R}_i, \vec{\xi}_i) = 0$

D'Alembert principle $\Leftrightarrow$ variational principle.

M – submanifold $M \subset \mathbb{R}^n$, $\vec{x}: \mathbb{R} \to M$ a curve with $\vec{x}(t_0) = \vec{x}_0$, $\vec{x}(t_1) = \vec{x}_1$

Def, $\vec{x}$ is called a conditional extremal of the function $\Phi = \int_{t_0}^{t_1} \left(\frac{\dot{\vec{x}}^2}{2} - U(\vec{x}) \right) dt$ If $\delta\Phi$ is equal to zero

when the variation consists of nearby curves joining $\vec{x}_0$ to $\vec{x}_1$ in M.

$\delta_M \Phi = 0 \quad \Leftrightarrow$ Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial L}{\partial \vec{q}}, \quad L = \frac{\dot{\vec{x}}^2}{2} - U(\vec{x}), \quad \vec{x} = \vec{x}(\vec{q})$$

Theorem: $\vec{x}$ is a conditional extremal iff

it satisfies the D'Alembert's equation

$$\left(\ddot{\vec{x}} + \frac{\partial U}{\partial \vec{x}}, \vec{\xi} \right) = 0, \quad \forall \xi \in TM_{\vec{x}}.$$

Lemma: $\vec{\xi}(t) \in TM_{\vec{x}(t)}$ with $\vec{\xi}(t) = 0$ for $t = t_0, t_1$

$\int_{t_0}^{t_1} \vec{f}(t) \cdot \vec{\xi}(t)\, dt = 0$

Then the field $\vec{f}(t)$ is perpendicular to M at $\vec{x}(t)$

proof; Idea: compare Φ on $\vec{x}(t)$ and $\vec{x}(t) + \vec{\xi}(t)$ with $\vec{\xi}(t_0) = \vec{\xi}(t_1) = 0$

$$\delta\Phi = \int_{t_0}^{t_1} \left(\dot{\vec{x}}\dot{\vec{\xi}} - \frac{\partial U}{\partial \vec{x}} \vec{\xi} \right) dt = -\int_{t_0}^{t_1} \left(\ddot{\vec{x}} + \frac{\partial U}{\partial \vec{x}} \right) \vec{\xi}\, dt.$$

$\Rightarrow \quad \delta_u \Phi = 0 \quad \Leftrightarrow \quad \int_{t_0}^{t_1} \left(\ddot{\vec{x}} + \frac{\partial U}{\partial \vec{x}}\right) \bar{\xi}\, dt = 0$

for all $\bar{\xi}(t) \in TM_{\vec{x}(t)}$ $\quad \Leftrightarrow \quad \left(\ddot{\vec{x}} + \frac{\partial U}{\partial \vec{x}}, \vec{\xi}\right) = 0$ by the Lemma.

Thm: $\vec{f} = -\frac{\partial U}{\partial x}$ The point $\vec{x}_0$ in M is an equilibrium position if and only if the force is orthogonal to the surface at $\vec{x}_0$: $(\vec{f}(\vec{x}_0), \vec{\xi}) = 0$ for all $\vec{\xi} \in TM_{\vec{x}_0}$

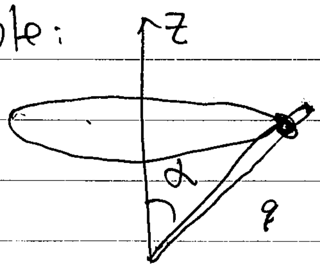
proof: $\left(\ddot{\vec{x}} + \frac{\partial U}{\partial x}, \vec{\xi}\right) = 0$ set $\ddot{\vec{x}} = 0$.

Def: $-m\ddot{\vec{x}}$ force of inertia.

Thm: Add force of inertia to the acting force. $\vec{x}$ becomes an equilibrium position

proof: $(-m\ddot{\vec{x}} + \bar{f}, \vec{\xi}) = 0$

Example:



$L = T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 r^2$

$r = q \sin\alpha$

Lagrange's eq.: $m\ddot{q} = m\omega^2 q \sin^2\alpha$

$(m\ddot{q}, \vec{\xi}) = 0$, $\vec{\xi} \perp \vec{r}$.

Oscillations 振动 (Small oscillations)

• Equilibrium positions

Def: $\vec{x}_0$ is the equilibrium position of the system
$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}), \quad \vec{x} \in \mathbb{R}^n$$
if $\vec{x}(t) = \vec{x}_0$ is a solution of this system. $\vec{f}(\vec{x}_0) = 0$

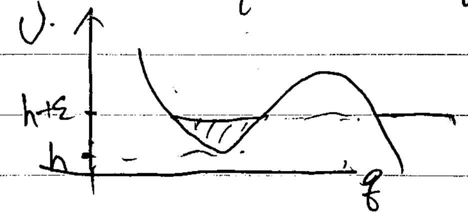
Thm: Consider a natural dynamical system with
$L(\vec{q}, \dot{\vec{q}}) = T - U$, where $T = \frac{1}{2} \sum a_{ij}(\vec{q})\, \dot{q}_i \dot{q}_j \geq 0$ and $U = U(\vec{q})$

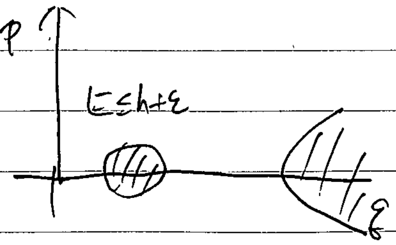
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \quad q = (q_1 \cdots q_n)$$

The point $\vec{q} = \vec{q}_0$, $\dot{\vec{q}} = \dot{\vec{q}}_0$ will be an equilibrium position iff
$\dot{\vec{q}}_0 = 0$, and $\vec{q}_0$ is a critical point of the potential energy
$$\left.\frac{\partial U}{\partial q}\right|_{\vec{q}_0} = 0$$

proof: $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial q} - \frac{\partial U}{\partial q}$$

Thm: If $\vec{q}_0$ is a strict local minimum of the potential U, then the equilibrium $\vec{q} = \vec{q}_0$ is stable.





Date

• Linearization of a differential equation

$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$, $\vec{f}(x) = A\vec{x} + R_2(\vec{x})$, $A = \frac{\partial f}{\partial \vec{x}}\Big|_0$, $R_2 = O(x^2)$

Linearization: $\frac{d\vec{y}}{dt} = A\vec{y}$ $\bar{y}$: linearized term.

$\bar{y}(t) = e^{At}\bar{y}(0)$, where $e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$

Thm: For ~~near~~ any $T>0$, $\varepsilon>0$, $\exists\, \delta>0$ s.t. if $|x(0)|<\delta$, then $|\vec{x}(t)-\bar{y}(t)|<\varepsilon\delta$ for all $t\in(0,T)$

▷ solution to the linearized system is a good apporximation.

• Linearization of a lagrangian system.

Thm: equilibrium position $\vec{q}=0$ it is sufficient to replace the kinetic energy $T=\frac{1}{2}a_{ij}(\vec{q})\,\dot{q}_i\dot{q}_j$ by its value at $\vec{q}=0$

$T_2 = \frac{1}{2}\sum a_{ij}\,\dot{q}_i\dot{q}_j$, $a_{ij}=a_{ij}(0)$

and replace the potential energy $U(\vec{q})$ by its quadratic part

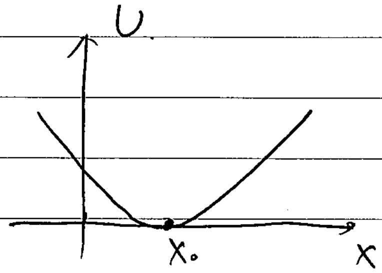
$U_2 = \frac{1}{2}\sum b_{ij}\,q_iq_j$ $b_{ij} = \frac{\partial^2 U}{\partial q_i\partial q_j}\Big|_{q=0}$

- The period τ of oscillations: $\tau_0 = \frac{2\pi}{\omega_0}$,

$\omega_0^2 = b/a$, $b = \left(\frac{\partial^2 U}{\partial q^2}\right)\Big|_{q=q_0}$, $a = a(q_0)$

proof: $T_2 = \frac{1}{2} a \dot{q}^2$ $U_2 = \frac{1}{2} b q^2 \Rightarrow \ddot{q} = -\omega_0^2 q$

- Small oscillations



Consider a system with:

$$T = \frac{1}{2} \underbrace{(A\dot{\vec{q}}, \dot{\vec{q}})}_{\text{positive definite}}, \quad U = \frac{1}{2}(B\vec{q}, \vec{q})$$

$Q = Cq$ s.t. $T = \frac{1}{2}\sum_{i=1}^{n} \dot{Q}_i^2$, $U = \frac{1}{2}\sum_{i=1}^{n} \lambda_i Q_i^2$

λ_i satisfies the characteristic equation:

$$\det|B - \lambda A| = 0$$

- Characteristic oscillations in the coordinates Q. $\ddot{Q}_i = -\lambda_i Q_i$

Case 1: $\lambda = \omega^2 > 0$ $Q = C_1 \cos\omega t + C_2 \sin\omega t$ ~ oscillation
Case 2: $\lambda = 0$ $Q = C_1 + C_2 t$ neutral equilibrium
Case 3: $\lambda = -k^2 < 0$ $Q = C_1 \cosh kt + C_2 \sinh kt$ — instability

Date . .

- Small oscillations $\Rightarrow$ direct product of n-dimensional small oscillations

- Suppose one of the eigenvalues is positive : $\lambda = \omega^2 > 0$

Then (1) can perform a small oscillation of the form

$$\vec{q}(t) = (C_1 \cos\omega t + C_2 \sin\omega t)\vec{\xi}.$$

where $\vec{\xi}$ is an eigenvector $B\vec{\xi} = \lambda A \vec{\xi}$

$\boxed{Q_i = C_1\cos\omega_i t + C_2 \sin\omega_i t \ , \ Q_j = 0 \ (j \neq i)}$ characteristic oscillation

- Every small oscillation is a sum of characteristic oscillations.

$$\vec{q} = e^{i\omega t}\vec{\xi}, \qquad \frac{d}{dt} A\vec{q} = -B\vec{q}$$

$$\Rightarrow (B - \omega^2 A)\vec{\xi} = 0$$

$$\vec{q}(t) = \mathrm{Re} \sum_{k=1}^{n} C_k e^{i\omega_k t} \vec{\xi}_k.$$